REGULAR QUANTUM DYNAMICS

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REGULAR QUANTUM DYNAMICS

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This work is dedicated to my mother,

Mary Rose Baugh Bacon.

Her support both moral and financial cannot be repaid.

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SUMMARY

Einstein described the problem of a quantum theory of space-time as “like trying to breathe in empty space.” Here we replace it by a more determined and well-posed problem. Recalling that quantum theory itself was born out of a regularization process, instead of setting out to quantize space-time we set out to regularize it. The problem is then to eliminate the Heisenberg singularity from quantum mechanics as economically as possible. We solve this problem here.

At the outset we explain the concepts of regular and singular groups and define the Heisenberg singularity. This singularity infests not only the theory of space-time, but also the Bose-Einstein statistics and the theory of the gauge fields and interactions. It is responsible for most of the infinities of present quantum field theory.

The key new conceptual step in our solution is to turn attention from observables to what actually enters into the Heisenberg dynamical equations. The dynamical equations do not relate observables but observable-valued-functions of time, which we call *dynamicals* for convenience. Observables and dynamicals have separate algebras and separate Lie algebras. Observables split the temporal (time-energy) variables from the system variables (position-momentum, in mechanics). Dynamicals do not. This reconception allows for the possibility of clock-system entanglement that is missing from the usual singular dynamics, and implied by the concept of quantum space-time.

To carry out our solution we first set up the algebraic theory of dynamicals. This defines a dynamical Lie algebra that includes both commutation relations and dynamical equations. Once this is done, almost any small change in its commutation relations gives every physical dynamical, including time, a discrete and bounded spectrum. Here we work this out for a toy system, the isotropic harmonic oscillator in any finite number of dimensions.

The result is a finite quantum dynamical theory, like that of the stationary states of a higher-dimensional spin. By regularizing the dynamical group, we automatically eliminate

the Heisenberg singularity. As a by-product we quantize space-time, but also momentum-energy and every other dynamical variable in the theory.

The regularization we carry out here requires changing exceptional commutation relations slightly to make them generic. The regularized commutation relations couples some new variables r called *regularizing variables* to the oscillator p, q. The regularized position and momenta generate a larger Lie algebra that is simple and so regular. The regular theory reduces to the singular one when the structure constants and the regularization variables approach their singular limits.

The time-dependent theory of the oscillator has the structure of the time-independent theory of an oscillator coupled to a clock. Quantizing time amounts mathematically to quantizing the clock, which we do when we regularize the group of the united system.

The main results are the following.

1. We regularize the canonical Lie algebra $\mathsf{h}(n)$ with n coordinates and n momenta by presenting it as a factor of a singular limit $\mathsf{sl}(n+1) \to \mathsf{sl}(n) \ltimes \mathsf{h}(n)$.

2. We regularize the dynamical or time-dependent harmonic oscillator Lie algebra $\boldsymbol{\mathcal{L}}_{\mathrm{HO}}(m)$, with $m+1$ coordinates including time, and $m+1$ momenta including energy, by presenting it as a factor of a singular limit $\mathsf{sl}(n+2) \to \mathsf{sl}(m+1) \supset \mathsf{h}(m+1)$.

The new theory implies corrections to quantum theory and special relativity that are small at low energies but which may be used to provide bounds on the new quantum constants that we have introduced.

CHAPTER 1

INTRODUCTION

Einstein described the project of a quantum theory of space-time as "trying to breathe in empty space." Nevertheless this problem has attracted attention since the early days of quantum field theory, primarily because one expects a quantum space-time theory to replace the divergent integrals over space-time that mar present theories by convergent sums. Einstein's lament expresses the fact that the problem is ill-posed, being vastly underdetermined. We lack a prescription corresponding to the Heisenberg quantization procedure of replacing Poisson brackets by commutators. In fact both special relativization and canonical quantization are special cases of a more general process of theory repair, called group regularization, and this process applies to space-time structure too. Group regularization, so to speak, is the air-line that Einstein wanted. It replaces the space-time quantization problem by one that is more determined, better posed, and more directly motivated: that of eliminating the Heisenberg singularity from quantum mechanics with the minimum of new hypotheses. Instead of setting out to quantize space-time we set out to regularize it, recalling that quantum theory itself was born out of a regularization process. We solve this problem for some simple toy physical systems here.

1.1 Singular and Regular Groups of Physics

When we construct a quantum theory from its underlying groups, we notice that some of the basic groups are regular (= stable, generic, robust,...), and some are singular (= unstable, exceptional, fragile, ...) in a sense defined below, and that the infinities of the theory seem to come from the singular groups. Special and general relativity and quantum theory substantially regularized some important groups and algebras of physics but several singular ones remain, especially the canonical (or " Heisenberg") Lie algebra $\mathsf{h}(r)$. This is the Lie algebra defined by the Lie product relations among r coordinates and their momenta, suitably scaled:

$$
\begin{aligned}
P_a \vartriangle Q^b &= \delta_a^b R \\
R \vartriangle P_a &= 0 \\
R \vartriangle Q^b &= 0 \qquad (1)
\end{aligned}
$$

where a and b range from 1 to $r+1$, and R is the central element of the Lie algebra usually identified with $\mathbf{1}$ in the operator representations.

Despite the name this Lie algebra is older than quantum theory, for it is the Lie algebra of the generators (q^k, p_k) of the canonical group and of the generators $(x^k\cdot, \frac{\partial}{\partial x^k})$ in differential calculus.

Extrapolating the process of group regularization would seem to lead to a finite quantum theory with a semi-simple group. Here we regularize the canonical Lie groups and apply the result to a theory that is completely regularized by this method, the harmonic oscillator in n dimensions, regularizing both the time-independent and time-dependent theories. This requires changing their commutation relations, of course, but the change can be as small as desired. It can therefore be adjusted to be compatible with past experimental data. It will necessarily make major changes in new experiments, for example, experiments at higher energy. Since it is no longer exactly true that $E = i\hbar d/dt$, we must also revise the separation of variables that leads from the time-dependent dynamical problem to the time-independent energy-level problem.

Groups are singular or regular in the following sense. The product of a Lie group is locally defined by a Lie algebra product $a \vartriangle b$ on a vector space $\mathcal{L}$, and is also called a structure tensor, being a mapping $[\mathcal{L}\otimes\mathcal{L}] \to \mathcal{L}$ and a tensor in the linear space $\mathcal{L}\otimes[\mathcal{L}\otimes\mathcal{L}]^*$, the brackets indicating skew-symmetry. The Lie algebra is properly the product operation $\vartriangle$, though it is traditionally identified by the same symbol as the space $\mathcal{L}$. The Lie products $\vartriangle$ on a given space $\mathcal{L}$ form a nonlinear submanifold $\mathsf{J}(\mathcal{L}) \subset \mathcal{L}\otimes[\mathcal{L}\otimes\mathcal{L}]^*$ defined by the Jacobi law. An algebra $\vartriangle$ is *regular* (= stable, robust, generic, ...) if for some neighborhood $N(\vartriangle) \subset \mathcal{J}(\mathcal{L})$, all products $\vartriangle' \in N(\vartriangle)$ are isomorphic to $\vartriangle$. Then small changes in the product $\vartriangle$ — that is, in the commutation relations — do not change the group, up to isomorphism. Otherwise $\mathcal{L}$ is *singular* (= unstable, fragile, exceptional, ...). Some algebra in any neighborhood of a singular algebra is non-isomorphic to the singular algebra; usually, in fact, almost all the algebras in any sufficiently small neighborhood.

We recapitulate some standard definitions for convenience: The derived Lie algebra $\partial\mathcal{L}$ of a Lie algebra $\mathcal{L}$ is the Lie subalgebra $\partial\mathcal{L} := \mathcal{L} \vartriangle \mathcal{L} \subset \mathcal{L}$, of Lie products. $\mathcal{L}$ is *solvable* if one of the iteratedly derived Lie algebras $\partial^n\mathcal{L}$ is trivial: $\exists\, n > 0 : \partial^n\mathcal{L} = \{0\}$. That is, $\mathcal{L}$ is solvable if by forming commutators of commutators of commutators ..., one always eventually reaches $\mathbf{0}$ in a bounded number of steps. The *radical* of a Lie algebra is its maximal solvable Lie subalgebra. A *semisimple* Lie algebra has trivial radical $\{0\}$. A Lie algebra and corresponding Lie group that is not semisimple we call *compound*. Compound Lie algebras have nontrivial radical and a singular Killing form.

Semisimple Lie groups are regular [12]. Often but not always, non-semisimple Lie groups are singular ; IGL_1 is one counterexample. Regular Killing forms make regular groups. The elements of the Lie algebra that are mapped to 0 by the Killing form we call the Killing kernel. The probability metric of a regular quantum theory derives from its Killing form.

Definition 1. *A homotopy φ of a Lie group G is a continuous mapping of the product space $\mathsf{G} \times \mathbf{T} \xrightarrow{\varphi} \boldsymbol{\mathcal{A}}$ where $\mathbf{T} \simeq \mathbb{R}$ is the space of the homotopy parameter τ, $\boldsymbol{\mathcal{A}}$ is a topological space with continuous product, and the image of φ for each fixed $\tau \in \mathbf{T}$ again defines a Lie group under the product for H.*

Typically the space $\boldsymbol{\mathcal{A}}$ is again a Lie group.

Lie group regularization is the process of continuously varying the product relations introducing additional terms in the generating relations. This amounts to a homotopy connecting the group to a non-isomorphic group. The inverse process is *group singularization* is defined by the inverse homotopy. We denote the singularization by $\mathsf{G}_{reg} \xrightarrow{\eth} \mathsf{G}_{sing}$ and the reversed regularization by $\mathsf{G}_{sing} \xrightarrow{\eth^*} \mathsf{G}_{reg}$.

Regularization "annihilates" the Killing kernel of the Lie algebra (reduces it to $\{0\}$) effecting a non-singular Killing form. The *g*roup contraction process of Inönu and Wigner is a special case of group singularization induced by a linear transformation of the algebra of the form [8]

$$\lambda \to e^{-\eta t}\lambda, \quad \eta \in \mathsf{L} \otimes \mathsf{L}^* : \eta^2 = \eta = \eta^*. \tag{2}$$

Inönü and Wigner relate their work to Segal's [12] Contraction produces a singular Lie algebra with abelian nil-radical. Singularization in general does not.

To carry out a group regularization of a singular theory, we must reconstruct the theory as follows:

1. Formulate the Lie algebra L underlying the theory.
2. Regularize that Lie algebra in a regular Lie algebra $\hat{\mathsf{L}}$.
3. Choose a sequence of finite-dimensional representation $\varrho_n : \hat{\mathsf{L}} \to \mathbf{End} V_n$ of $\hat{\mathsf{L}}$ by operators on quadratic spaces V_n with $n \to \infty$.
4. Choose a subspace of each V_n within which the regularization variables that are central in the Heisenberg limit approach their singular limits.
5. Represent the physical operations of the theory in one representation of the sequence ϱ_n.

When we approximate a circle by a tangent, they agree only on a small part of each. Similarly, the regular and singular theories approximate each other only on a small part of both representation spaces, chosen in step 5.

Usually the invariant quadratic form of the representation space V is indefinite. Then physical operators for each frame are induced on a subspace of V with positive definite-form.

The project encounters problems, including some strong predictions. To enumerate some of these problems:

1. There are several unstable groups to be regularized in the present physics.
2. Our regularization makes substantial conceptual changes in physics, such as the quantization of time and space and the relativization of the system-laboratory interface.
3. Our regularization introduces new dynamical variables that must be found in nature.

4. Our regularization requires that these regularization variables be frozen out in the singularization process, as by a spontaneous symmetry-breaking.

5. Lie groups themselves are unstable within the broader theory of quantum groups, in which Segal was a pioneer.

6. It is not immediately obvious that regular groups make a finite theory. We must also consider the selection of the irreducible representations defining the physical systems.

None of these difficulties seem insurmountable, and some open promising vistas. Fo example, it seems likely to us that regularization variables are the seeds of gauge dynamical variables.

Here are three standard examples of group regularization for orientation.

1.2 The Galileo and Lorentz Group

The transition from the Galileo group to the Lorentz group is a group regularization.

The Lorentz group is regular. Its Killing form in the standard basis $\ell_{\mu\nu}$ is the regular quadratic form (3) of signature 0 in 6 dimensions.

$$K^{(2)}(\lambda^{\mu\nu}\ell_{\mu\nu}) = \frac{1}{c^2}\left[(\lambda^{01})^2 + (\lambda^{02})^2 + (\ell^{03})^2\right] - (\lambda^{12})^2 - (\lambda^{23})^2 - (\lambda^{31})^2 \tag{3}$$

Its Galilean limit $c \to \infty$ is singular; its Killing kernel is the abelian subalgebra of Galilean boosts.

Before Einstein's regularization, space-time was a bundle of space fibers on an absolute time base. In this special example this induced a bundle of spatial orientations over a base of velocity frames. The Lorentz egularization unified this base and fiber. Eliminating the Killing kernel eliminated an associated absolute. This case of group regularization is an inverse to a group contraction.

1.3 The Poincaré and deSitter Group

The regularization resulting in the Lorentz group is part of a larger more complete regularization. The larger group of Galilean physics is 10-dimensional when the normal subgroup of space and time translations is included. Regulatization of the Galileo subgroup results in the 10-dimensional Poincaré group $\mathsf{ISO}(3,1)$ of Lorentz transformations and space-time translations. The killing kernel is the normal abelian sub-algebra of space-time translators $\{P_\mu\}$.

By introducing a small non-commutivity

$$[P_\mu, P_\nu] = \kappa\lambda_{\mu\nu} \tag{4}$$

the Poincaré group is further regularized to one of the simple deSitter groups $\mathsf{SO}(4,1)$ or $\mathsf{SO}(3,2)$. The physical effect is that of introducing a cosmological constant into Einstein's vacuum equations.

1.4 Canonical Lie Group

The transition from classical mechanics to quantum is a partial group regularization.

The classical commutativity relation $[q,p]=0$ defines a singular algebra with $\triangle = 0$. The entire algebra is its radical and its Killing kernel. The Heisenberg commutation relation

$$[q,p] = \hbar i\mathbf{1} \tag{5}$$

defines the canonical Lie group $\mathsf{H}(1)$. This has the same radical and Killing kernel as the classical commutative group. Nevertheless it is stable against small variations in $\hbar$ and therefore is less singular than the commutative algebra. The nth canonical Lie group $\mathsf{H}(n)$ is just the $n$the tensor power $\mathsf{H}(1)^{\otimes n}$. The limit $\hbar \to 0$ defines a group contraction.

1.5 Segal group

The transition from the canonical Lie group to an orthogonal group proposed by Segal is a group regularization.

The canonical Lie algebra is not yet regular. Segal proposed the regular relations (up to notation)

$$[\widehat{q},\widehat{p}] = \hbar i\widehat{r}, \quad [\widehat{r},\widehat{q}] = \hbar' i\widehat{p}, \quad [\widehat{r},\widehat{p}] = -\hbar'' i\widehat{q}, \tag{6}$$

with $\widehat{r} \approx 1$ in the domain where the Heisenberg quantum theory works [12, 2, 10, 16, 17, 18, 14, 13]. This morphs the canonical Lie group $\mathsf{H}(1) \to \mathsf{SO}(2,1)$. For physical reasons we use $\mathsf{SO}(3)$ instead. It introduces a new dynamical variable $\widehat{r}$ (Segal's Y_3) that we call the *regularization variable*. For $\mathsf{SO}(3)$, $-1 \leq \widehat{r} \leq 1$. This example illustrates two of the problems we have enumerated above: the new regularization variable $\widehat{r}$ and two fundamental constants $\hbar'$, $\overline{\prime}'$ have to be found experimentally.

The stationary linear harmonic oscillator is based on the unstable operation group $\mathsf{H}(1)$. The dynamical theory is based on $\mathsf{H}(1) \ltimes \mathsf{H}(1)$, which includes the operation group of both the system and the clock. Covariant theories use the groups $\mathsf{H}(4)$ and $\mathsf{H}(1) \ltimes \mathsf{H}(n)$.

1.6 Stationary and Dynamical Systems

The stationary (time-independent) quantum oscillator in one dimension has already been regularized and examined [13]. Here we regularize the stationary and the dynamical (time-dependent) n-dimensional oscillator. In the process we also regularize the Heisenberg group $\mathsf{H}(n)$.

It is important to recognize that there are two operation algebras in ordinary quantum dynamics, one with time dependence and one without.

The usual "algebra of observables" or operation algebra of the system is the lesser of the two, and the other, which we call the dynamical algebra of the system, is isomorphic to the operation algebra of the system-and-clock. Ordinarily we consider as observables only operators in the system-and-clock operation algebra that commute with time. The dynamical equations relate system-and-clock operations to each other. We reduce these to system operations by stopping the clock, as by setting $t = 0$, to the extent that a quantum clock can be stopped. In the singular limit, a projection on the subspace with $t = 0$ converts hermitian dynamicals into hermitian observables and so does a trace over the operator E. We can equivalently base the concept of observable on either time or energy. Both of these concepts are available in the regular theory as well as the singular, but while they agree in the singular theory, they differ in the regular theory. Apparently there is a doubling of the observable concept with a corresponding complementarity principle. Such a doubling of physical concepts under relativization occurred in special relativity, as a doubling of the time concept, and we learned there how to deal appropriately with both proper time and coordinate time. Here we may have to learn how to deal appropriately with two kinds of observable, one resulting from fixing time and one from averaging over energy.

The singularization that is the inverse of this group regularization process is not a contraction but a more general homotopy. One may accomplish a singularization of a Lie group L by a linear operator $L : \mathsf{L} \to \mathsf{L}$ which we call the singularizer. Our method specifically defines the singularizer as an inner derivation $L \equiv \triangle\eta$ of a larger enveloping Lie algebra $\eta \in \mathsf{L}' \supset \mathsf{L}$.

We call the number of points in the spectrum of the singularizer the *order* of the associated singularization. Contractions have order 2 and the Segal singularization of the canonical group $\mathsf{H}(1)$ has order 3.

Our work consists of the following stages:

1. Formulate the dynamical equations as the relations of a dynamical Lie algebra.

2. Regularize the dynamical Lie algebra with as few new variables as possible.

3. Represent the regular Lie algebra in a finite dimensional state-vector space.

4. Select a subspace of the representation space in which the Heisenberg dynamical relations hold as exactly as past experiments require.

We find the following results, among other similar ones:

One regularization of $\mathsf{H}(n;\mathbb{C})$ is an inverse singularization $\mathsf{SL}(n+1) \leftarrow \mathsf{SL}(n) \ltimes \mathsf{H}(n;\mathbb{C})$ within the simple group $\mathsf{SL}(n+1)$. This regularization introduces n^2 regularization variables corresponding to the n^2-1 generators of the Lie group $\mathsf{SL}(n;\mathbb{C})$ and the single generator in the center of $\mathsf{H}(r)$.

$\mathsf{H}(2;\mathbb{C})$, however, has two other regularizations, within the semi-simple groups $\mathsf{SO}(4)$ and $\mathsf{SO}(3,1)$ [13], still with $n^2 = 4$ regularization variables.

For example, $\mathsf{H}(3)$ can be regularized by discovering an $\mathsf{SU}(3)$ in nature and providing a homotopy of $\mathsf{SU}(4) \to \mathsf{SU}(3) \times \mathsf{H}(3)$, a semidirect product. No smaller group than $\mathsf{SU}(3)$ suffices for this, and no larger one is necessary. In the singular limit the $\mathsf{SU}(3)$ variables may freeze out.

CHAPTER 2

INHOMOGENOUS LIE GROUPS AND ALGEBRAS

2.1 Introduction

In the next section we construct a singular homotopy

$$\mathsf{sl}(r+1) \to \mathsf{sl}(r+1) \ltimes \mathsf{h}(r). \tag{7}$$

of the special linear algebra $\mathsf{sl}(r+1)$ that results in a *canonical special linear* Lie algebra $\mathsf{hsl}(r)$ defined by the Lie product relations

$$\begin{aligned} P_a \bigtriangleup Q^b &= \delta^b_a R \\ R \bigtriangleup P_a &= 0 \\ R \bigtriangleup Q^b &= 0 \end{aligned} \tag{8}$$

where a and b range from 1 to $r+1$.

$\mathsf{hsl}(r)$ extends $\mathsf{isl}(r)$, which includes the r commuting coordinates in p^r, by its dual, the r commuting momenta, subject to the canonical commutation relations. Our singularization of $\mathsf{sl}(r+1) \to \mathsf{hsl}(r)$ is analogous to the contraction of $\mathsf{so}(n+1) \to \mathsf{iso}(n)$. The nilradical of the inhomogeneous algebra $\mathsf{iso}(n)$, however, is abelian. This is true of all contractions. The nil-radical of the singular algebra $\mathsf{hsl}(r)$ is not abelian, being the canonical Lie algebra h_r itself. Therefore h_r is not a contraction of a simple group but a more general kind of singularization.

The quotient algebra $\mathsf{hsl}(r)/\mathsf{h}(r)$ is the special linear Lie algebra $\mathsf{sl}(r)$ and thus we may express $\mathsf{hsl}(r)$ as the semi-direct sum

$$\mathsf{hsl}(r) = \mathsf{sl}(r) \ltimes \mathsf{h}(r) \tag{9}$$

This singularization has important physical application in quantum mechanics, where the classic Heisenberg relations originated.

Then we show that this induces singular homotopies of the classical Lie algebras of Table 1, taken over any of the classical fields $\mathbb{R}$,, $\mathbb{C}$, $\mathbb{H}$. The algebra $\mathsf{p}(r)$ is the r dimensional abelian Lie algebra.

Table 1: Induced Singular Homotopies

Homogenous Algebra	Inhomogenous Algebra	Nil-Radical
Linear $\mathsf{sl}(r+1)$	Canon. Linear $\mathsf{hsl}(r) = \mathsf{sl}(r) \ltimes \mathsf{h}(r)$	Canonical $\mathsf{h}(r)$
Unitary $\mathsf{su}(r+1)$	Canon. Unitary $\mathsf{hsu}(r) = \mathsf{su}(r) \ltimes \mathsf{h}(r)$	Canonical $\mathsf{h}(r)$
Orthogonal $\mathsf{so}(r+1)$	Inh. Orthogonal $\mathsf{iso}(r) = \mathsf{so}(r) \ltimes \mathsf{p}(r)$	Translation $\mathsf{p}(r)$
Symplectic $\mathsf{sp}(r)$	Inh. Symplectic $\mathsf{isp} = \mathsf{sp}(r) \ltimes \mathsf{p}(r)$	Translation $\mathsf{p}(r)$

Key:
"Inh." indicates an extension by a translation algebra $\mathsf{p}(r)$.
"Canon." indicates an extension by a canonical Lie algebra $\mathsf{h}(r)$.

The classical groups are homogenous linear transformations. We define inhomogenous counterparts for them as in Table 2.

Table 2: Groups Over Spaces

Space	Homogenous Group	Inhomogenous Group	Singular Group
Linear $\mathbf{V}$	Linear $\mathsf{SL}(\mathbf{V})$	Canon. Linear $\mathsf{HSL} = \mathsf{SL} \ltimes \mathsf{LH}$	Canonical $\mathsf{H}(\mathbf{V})$
Hermitian $\mathbf{H}$	Unitary $\mathsf{SU}(\mathbf{H})$	Canon. Unitary $\mathsf{HSU} = \mathsf{SU} \ltimes \mathsf{UH}$	Canonical $\mathsf{H}(\mathbf{H})$
Quadratic $\mathbf{X}$	Orthogonal $\mathsf{SO}(\mathbf{X})$	Inh. Orthogonal $\mathsf{ISO} = \mathsf{SO} \ltimes \mathsf{P}$	Translation $\mathsf{P}(\mathbf{X})$
Symplectic $\mathbf{W}$	Symplectic $\mathsf{Sp}(\mathbf{W})$	Inh. Symplectic $\mathsf{ISp} = \mathsf{Sp} \ltimes \mathsf{P}$	Translation $\mathsf{P}(\mathbf{W})$

These inhomogenous groups are singularizations of the corresponding homogenous cases that preserve a subgroup embedding. We designate such singularizations by, for example,

$$\mathsf{SL}_{n+1} \xrightarrow[\mathsf{SL}_n \subset \mathsf{SL}_{n+1}]{\eth} \mathsf{HSL}_n, \tag{10}$$

2.2 *The Heisenberg Singularity*

We begin our main construction by defining a non-standard basis for the rank r Lie algebra $\mathsf{sl}(r+2)$. For familiarity we express this basis in terms of a vector representation in (11), using a real or complex $r+2$ dimensional space $\mathbf{V}$. We introduce a vector basis $\psi^k \in \mathbf{V}$ and the dual basis $\psi^*_k \in \mathbf{V}^*$ and indices indices $i, j, k = 0, 1, 2, \cdots, r+1$. Then we define algebra generators

$$\lambda^i_j \simeq \psi^i \otimes \psi^*_j - \frac{1}{r+2}\delta^i_j \mathbb{1}. \tag{11}$$

This Lie basis is in fact over-complete, with vanishing diagonal sum

$$\lambda^j_j = \mathbf{0}. \tag{12}$$

To construct an independent basis we take (12) to define the Lie algebra element λ^0_0 in terms of the other elements, which form an independent basis.

This enables us to express the Lie structure in the following simple form:

$$\lambda^i_j \vartriangle \lambda^m_n = \delta^m_j \lambda^i_n - \delta^i_n \lambda^m_j \tag{13}$$

We now define an initial $\mathsf{sl}(r+1)$ subalgebra spanned by the generators

$$\begin{aligned} \Lambda^b_a &= \lambda^b_a \\ \Lambda^1_a = \tilde{P}_a &= \lambda^1_a \\ \Lambda^b_1 = \tilde{Q}^b &= \lambda^b_1 \\ \Lambda^1_1 = \tilde{R} &= \lambda^1_1, \end{aligned} \tag{14}$$

$a, b = 2, 3, \ldots, r+1$, excluding 0 and 1. We have labeled the elements in anticipation of the singular limit to be constructed. We have not utilized any of the generators with zero index. Due to the condition (12), however, λ^0_0 is an element of this subalgebra.

Definition 2 (Segal). *A* motion *of a Lie algebra* L *is a one-parameter group of automorphisms of* L*; and analogously for Lie groups.*

We call *inner motion* a motion composed of inner automorphisms.

We now define the inner motion $g(s) = e^{s\vartriangle\eta}$ of $\mathsf{SL}(r+2)$ generated by the adjoint action $\vartriangle\eta : \xi \mapsto \eta \vartriangle \xi$ of element $\eta = \lambda^0_1 + \lambda^1_0$. As an automorphism of of the Lie algabra $\mathsf{sl}(r+2)$,

$g(s)$ will map subalgebras to distinct but isomorphic subalgebras. We consider, however, the *projective* action of $g(s)$ on the $\mathsf{sl}(r+1)$ subalgebra defined by (14), and its limit as $s \mapsto \infty$.

The adjoint action $\triangle\eta$ on the initial basis is

$$\begin{aligned} \eta \triangle \omega_a^b &= 0 \\ \eta \triangle \tilde{P}_a &= \lambda_a^0 \\ \eta \triangle \tilde{Q}^b &= -\lambda_0^b \\ \eta \triangle \tilde{R} &= \lambda_1^0 - \lambda_0^1 =: Y. \end{aligned} \tag{15}$$

The higher-order action of $\triangle\eta$ is

$$\begin{aligned} \eta \triangle \lambda_a^0 &= \tilde{P}_a \\ \eta \triangle \lambda_0^b &= -\tilde{Q}^b \\ \eta \triangle Y &= 2(\lambda_1^1 - \lambda_0^0) =: 2Z \\ \eta \triangle Z &= 2(\lambda_1^0 - \lambda_0^1) = 2Y \end{aligned} \tag{16}$$

These give us the hyperbolic transformation:

$$\begin{aligned} g(s)\tilde{P}_a &= \cosh(s)\tilde{P}_a(0) + \sinh(s)\lambda_a^0 \\ g(s)\tilde{Q}^b &= \cosh(s)\tilde{Q}^b(0) - \sinh(s)\lambda_0^b \\ g(s)\tilde{R} &= U + \cosh(2s)Z + \sinh(2s)Y \end{aligned} \tag{17}$$

where $U = \lambda_0^0 + \lambda_1^1$. In the limit $s \to \infty$ we then have

$$\begin{aligned} g(s)\tilde{P}_a &\to e^s P_a \\ g(s)\tilde{Q}^b &\to e^s Q^b \\ g(s)\tilde{R} &\to e^{2s} R \end{aligned} \tag{18}$$

where

$$\begin{aligned} P_a &= \lambda_a^1 + \lambda_a^0 \\ Q^b &= \lambda_1^b - \lambda_0^b \\ R &= \lambda_1^1 + \lambda_0^1 - \lambda_1^0 - \lambda_0^0 \end{aligned} \tag{19}$$

These limiting generators then obey the Heisenberg commutation relations (1).

Note that the $\mathsf{sl}(r)$ subalgebra generated by Λ^a_b has remained invariant throughout the process $g(s)$.

2.3 The Regularizing Expansion

To regularize $\mathsf{hsl}(r)$ we then reverse the limit (18):

$$\begin{aligned} P_a &\rightarrow e^{-s}[g(s)\tilde{P}_a] \equiv e^{-s}\hat{P}_a \\ Q^b &\rightarrow e^{-s}[g(s)\tilde{Q}^b] \equiv e^{-s}\hat{Q}^b \\ R &\rightarrow e^{-2s}[g(s)\tilde{R}] \equiv e^{-2s}\hat{R} \end{aligned} \tag{20}$$

where $0 \leq s < \infty$. Since the action of $g(s)$ is an isomorphism of $\mathsf{sl}(r+1)$ subgroups of $\mathsf{sl}(r+2)$, we can replace our original independent basis by its image under $g(s)$.

$$\begin{aligned} P_a &\rightarrow e^{-s}\hat{\Lambda}^1_a \\ Q^b &\rightarrow e^{-s}\hat{\Lambda}^b_1 \\ R &\rightarrow e^{-2s}\hat{\Lambda}^1_1 \\ \Lambda^a_b &\equiv \hat{\Lambda}^a_b \end{aligned} \tag{21}$$

The replaced $\hat{\Lambda}$'s have identical Lie relations as the original Λ's. The regularized Heisenberg relations are then

$$\begin{aligned} \hat{P}_a \vartriangle \hat{Q}^b = \sigma^2\hat{\Lambda}^1_a \vartriangle \hat{\Lambda}^b_1 &= \delta^b_a\hat{R} - \sigma^2\hat{\Lambda}^b_a \equiv \hat{R}^b_a \\ \hat{R} \vartriangle \hat{P}_a = \sigma^3\hat{\Lambda}^1_1 \vartriangle \hat{\Lambda}^1_a &= \sigma^2\hat{P}_a \\ \hat{R} \vartriangle \hat{Q}^b = \sigma^3\hat{\Lambda}^1_1 \vartriangle \hat{\Lambda}^b_1 &= -\sigma^2\hat{Q}^b. \end{aligned} \tag{22}$$

where $\sigma = e^{-s}$ is an arbitrary positive constant.

This regularizes the canonical Lie algebra $\mathsf{h}(r+1)$ introducing r^2 regularization variables $\hat{R}$ and $\hat{R}^a_b = \hat{R} - \sigma^2\hat{\Lambda}^b_a$ spanning the $\mathsf{gl}(r)$ subalgebra of $\mathsf{sl}(r+1)$.

CHAPTER 3

REGULARIZED OSCILLATOR DYNAMICS

3.1 Introduction

Most discussions of quantum theory concentrate on the "algebra of observables." Since almost none of its elements represent observables, but all of them represent operations, we call this algebra the *o*peration algebra of the system. Underlying it can be a smaller "seed" Lie algebra of basic operations, as the three-dimensional canonical Lie algebra $\mathsf{h1}$ with three generators $q, p, 1$ underlies the infinite-dimensional operation algebra of a harmonic oscillator.

The dynamical law concerns a different and still larger algebra, the algebra of observable-valued functions of time. which we call the *dynamical algebra* for short. The dynamical equation are relations in this algebra. It too may have a much smaller seed, like the dynamical canonical Lie algebra constructed below.

Dynamicals act on dynamical wave-functions of the form $\psi(q,t)$, which make up a vector space of the form $S \otimes T$, with one factor for the system and one for time. Therefore the dynamical algebra too has a product structure $S \otimes T$, where now S is the algebra of observables of the system, and T is the algebra of real functions of time. We call the product $U = S \otimes T$. Since time is what clocks read, we identify T with a clock. Then the dynamical algebra U of the system is exactly the operator algebra of the system-and-clock. We quantize time by quantizing a clock, leaving for later the question of which clock we take and how the correlation arises between the clock and the system, in order to accomplish our primary goal of a regular theory. Using the algebra U allows for the new possibility of entanglement between different values of time, implicit in the concept of quantum time.

In order to be able to write the Heisenberg equations of motion within this algebra, we include in T the operators both of time t and "energy" $E := i\hbar\partial_t$. S contributes q and $p = -i\hbar\partial_q$. Time t is now represented by a spatial coordinate of the clock, and E is the canonical momentum of t.

In [13] the one-dimensional time-independent harmonic oscillator theory was fully regularized. The Lie algebra regularization utilized was:

$$\mathsf{h}(1) \xrightarrow{\eth^*} \mathsf{so}(3) \tag{23}$$

This theory relies on an isomorphism unique to the rank-one simple Lie groups, $\mathsf{su}(2) \simeq \mathsf{so}(3)$. There is a corresponding isomorphism for the singular cases:

$$\mathsf{h}(1) \simeq \mathsf{iso}(2) \equiv \mathsf{so}(2) \ltimes \mathsf{p}(2) \tag{24}$$

where $\mathsf{p}(r)$ is r-dimensional abelian Lie algebra.

From Chapter II we see the higher dimensional generalizations

$$\begin{array}{rcl} \mathsf{so}(r+2) & \overset{\eth}{\to} & \mathsf{so}(r+1) \ltimes \mathsf{p}(r+1) \\ \mathsf{su}(r+1) & \overset{\eth}{\to} & \mathsf{su}(r) \ltimes \mathsf{h}(r) \end{array} \tag{25}$$

These two regularizations coincide only for the special case $r = 2$.

3.2 Harmonic Oscillator Dynamics

The N-dimensional isotropic harmonic oscillator of mass M and stiffness $K = \omega^2 M$ has a dynamical algebra generated by spatial operators q^k, p_k which are system observables ($k \in N$) and operators t, E which are not system observables but observables of a larger composite of system and clock. E and t do not commute but obey the Heisenberg relation

$$[E, t] = i\hbar\mathbf{1} \tag{26}$$

because the clock is a quantum system too. Since the operator identity $\mathbf{1}$ is central, Heisenberg's commutation relations are singular. It is this singularity that we now eliminate as we regularize the dynamics of the oscillator.

The Heisenberg commutation relations and oscillator dynamical equations combined are

$$\begin{aligned} [q^k, p_j] &= +i\hbar\delta^k_j \mathbf{1}, \\ [E, t] &= i\hbar\mathbf{1}, \\ [E, q^j] &= -i\hbar M^{-1} p_j, \\ [E, p_j] &= +i\hbar\omega^2 M q^j. \end{aligned} \tag{27}$$

for $j, k = 1, 2 \ldots N$, all other pairwise commutators vanishing. These relations define the harmonic oscillator's dynamic Lie algebra

$$\boldsymbol{\mathcal{L}}_{\text{HO}}(N) := \mathsf{h}(1) \ltimes \mathsf{h}(N) \tag{28}$$

This is the algebra to be regularized.

3.3 Extending The Dynamic Algebra

We adjoin to $\boldsymbol{\mathcal{L}}_{\text{HO}}$ the operator $p^0 = -p_0$ which commutes with the p_k and q^k and E of the Harmonic oscillator but obeying the canonical commutation relations as E with t. We may then extend $\boldsymbol{\mathcal{L}}_{\text{HO}}$ to a larger Lie algebra $\boldsymbol{\mathcal{L}}'_{\text{HO}}$ generated by $E, p^0, q^0 = t, p_k$ and q^k.

The subalgebra generated just by the p_a and q^a for $a = 0, 1, \cdots$ then is $\mathsf{h}(N+1)$. The harmonic oscillator Hamiltonian H is then defined as the difference

$$H = E - p^0 = E + p_0 \tag{29}$$

By assumption $[E, t] = [p^0, t]$ and thus

$$[H, t] = \mathbf{0}. \tag{30}$$

With a particular index convention, $p_0 = -p^0$ we re-express the extended dynamic relations as

$$\begin{aligned} [q^a, p_a] &= +i\hbar\delta^k_j \mathbf{1} \\ [H, q^j] &= -i\hbar M^{-1} p_j, \\ [H, p_j] &= +i\hbar\omega^2 M q^j, \end{aligned} \tag{31}$$

which are those of $\mathcal{L}'_{\rm HO}$. We then have an unambiguous definition

$$\mathcal{L}'_{\rm HO} = \mathsf{gl}(1) \ltimes \mathsf{h}(n+1) \tag{32}$$

The quotient algebra $\mathsf{gl}(1)$ is the Lie agebra generated by the Hamiltonian (plus an arbitrary element of $\mathsf{h}(n+1)$) and its semi-direct sum with $\mathsf{h}(n+1)$ is expressed by (31).

In preparation to regularizing the Lie algebra we shall extend further in anticipation of applying the results from Chapter II. We embed the extended Lie algebra $\mathcal{L}'_{\rm HO}$ via

$$\mathcal{L}'_{\rm HO} = \mathsf{gl}(1) \ltimes \mathsf{h}(N+1) \xrightarrow{\subset} \mathsf{sl}(N+1) \ltimes \mathsf{h}(N+1). \tag{33}$$

The motivation for this extension is the mathematical technique developed earlier.

3.4 Regularization of the Dynamic Lie Algebra

It is a relatively simple matter to regularize $\mathcal{L}_{\rm HO}(N)$ leaving G invariant using the results of Chapter II. We begin by introducing a change of basis for the Lie algebra.

$$\begin{aligned} P_a &= p_a - i\omega M q^a, \\ Q^a &= p_a + i\omega M q^a, \\ R &= 2\hbar\omega M\mathbf{1}, \end{aligned} \tag{34}$$

for $a = 0, 1, \cdots, N$.

The choice of basis places the commutation relations (27) then take the simple form

$$[P_a, Q^b] = \delta_a^b R \tag{35}$$

where a and b range from 0 to N. This we identify with the form (14) of the previous chapter.

In addition the new basis is an eigen-basis of the Hamiltonian under the adjoint action.

$$\begin{aligned} [H, P_k] &= +\hbar\omega P_k, \\ [H, Q^k] &= -\hbar\omega Q^k, \\ [H, P_0] &= 0 = [H, Q^0]. \end{aligned} \tag{36}$$

We may then express the Hamiltonian in terms of our general $\mathsf{sl}(N+2)$ basis given (14).

$$H = -\hbar\omega \left[\sum_{k=1}^{N} \Lambda_k^k \right] \tag{37}$$

With this normalizing transformation equation (34) translate directly (21) where $r = N+1$ with the slight modification of transposing the index values 0 and 1 with the values N and $N+1$ respectively.

$$\begin{aligned} \hat{P}_a &= \sigma \hat{\Lambda}_a^{N+1} \\ \hat{Q}^b &= \sigma \hat{\Lambda}_{N+1}^b \\ \hat{R} &= \sigma^2 \hat{\Lambda}_{N+1}^{N+1} \end{aligned} \tag{38}$$

where a, b range from 0 to N and the $\hat{\Lambda}_b^a$ are as in (35). Regularization does not change the form of the Hamiltonian.

$$\hat{H} = -\hbar\omega \left[\sum_{k=1}^{N} \hat{\Lambda}_k^k \right] \tag{39}$$

The regularized product relations corresponding to (35) are

$$\begin{aligned} [\hat{P}_a, \hat{Q}^b] &= \delta_a^b \hat{R} - \sigma^2 \hat{\Lambda}_a^b \\ [\hat{R}, \hat{P}_a] &= +\sigma^2 \hat{P}_a \\ [\hat{R}, \hat{Q}^a] &= -\sigma^2 \hat{Q}^a \end{aligned} \tag{40}$$

where a ranges from 0 to N. Additional commutators between elements $\{\hat{P}_a, \hat{Q}^b, \hat{R}\}$ are all zero. The commutators with respect to $\hat{\Lambda}_b^a$ are unchanged.

The inverse of (34) is

$$
\begin{aligned}
\hat{p}_a &= (1/2)(\hat{P}_a + \hat{Q}^a) = (\sigma/2)[\hat{\Lambda}_a^{N+1} + \hat{\Lambda}_{N+1}^a], \\
\hat{q}^a &= (i/2\omega M)(\hat{P}_a - \hat{Q}^a) = (i\sigma/2\omega M)[\hat{\Lambda}_a^{N+1} - \hat{\Lambda}_{N+1}^a], \\
\hat{r} &= (1/2\hbar\omega M)\hat{R} = (\sigma^2/2\hbar\omega M)\hat{\Lambda}_{N+1}^{N+1}.
\end{aligned} \tag{41}
$$

We define the new notation

$$
\begin{aligned}
X_b^a &= \frac{1}{2}[\hat{\Lambda}_b^a + \hat{\Lambda}_a^b] = X_a^b \\
Y_b^a &= \frac{i}{2}[\hat{\Lambda}_b^a - \hat{\Lambda}_a^b] = -Y_a^b.
\end{aligned} \tag{42}
$$

Then the regularized $\hat{p}_a$ and $\hat{q}^a$ have the manifestly Hermitian form

$$
\begin{aligned}
\hat{p}_a &= \sigma X_a^{N+1} \\
\hat{q}^a &= (\sigma/\omega M) Y_a^{N+1} \\
\hat{r} &= (\sigma^2/\hbar\omega M) X_{N+1}^{N+1}
\end{aligned} \tag{43}
$$

for $a = 0, 1, 2, \cdots N$. The regularized commutation relations take the following form.

$$
\begin{aligned}
[\hat{q}^a, \hat{p}_b] &= +i\hbar\delta_b^a \hat{r} - (i\sigma^2/2\omega M) X_b^a \\
[\hat{r}, \hat{p}_b] &= -(i\sigma^2/\hbar) q^b \\
[\hat{r}, \hat{q}^a] &= +(i\sigma^2/\hbar\omega^2 M^2) p_a \\
[\hat{p}_a, \hat{p}_b] &= +(i\sigma^2/2) Y_a^b \\
[\hat{q}^a, \hat{q}^b] &= +(i\sigma^2/2\omega^2 M^2) Y_a^b
\end{aligned} \tag{44}
$$

for a, b again ranging from 0 to N.

In the notation defined by (42) the Hamiltonian takes the form

$$
\hat{H} = -\hbar\omega \left[\sum_{k=1}^{N} \hat{X}_k^k \right]. \tag{45}
$$

The expanded relations with respect to $\hat{H}$ are unchanged.

$$
\begin{aligned}
[\hat{H}, \hat{q}^j] &= -i\hbar M^{-1} \hat{p}_j, \\
[\hat{H}, \hat{p}_j] &= +i\hbar\omega^2 M \hat{q}^j, \\
[\hat{H}, \hat{p}_0] &= [\hat{H}, \hat{t}] = 0.
\end{aligned} \tag{46}
$$

We consider now the regularized time-energy relation.

$$[\hat{t}, \hat{E}] = [\hat{t}, \hat{p}_0] = +i\hbar\hat{r} - (i\sigma^2/2\omega M)X_0^0 \tag{47}$$

The time-energy Heisenberg relation for the singular theory expresses the energy E as the generator of parametric clock-system evolution $U(\tau) = e^{-i\hbar\tau E}$ such that the clock variable t is translated with unit parameter velocity

$$\frac{1}{i\hbar}[t, E] \equiv \frac{\mathsf{d}t}{\mathsf{d}\tau} = 1 \tag{48}$$

Thus the parameter is idendified with clock variable, $\tau \equiv t$.

In the regularization we abandon this identification however we may still interpret the time-energy commutator relation as the parametric velocity $\frac{\mathsf{d}\hat{t}}{\mathsf{d}\tau}$ of the regularized clock variable. This expresses the correlation of the system clock with a classical external clock τ where we now interpret $\hat{E}$ as the Hamiltonian of the system-clock composite with respect to the τ clock.

By this reasoning then from (47) we have:

$$\frac{\mathsf{d}}{\mathsf{d}\tau}\hat{t} = \hat{r} - (\sigma^2/2\hbar\omega M)X_0^0, \tag{49}$$

and the second τ derivative yields

$$\begin{aligned}
[\frac{\mathsf{d}}{\mathsf{d}\tau}]^2\hat{t} &= \frac{1}{i\hbar}[[\hat{t}, \hat{E}], \hat{E}] \\
&= (1/i\hbar)[\hat{r}, \hat{p}_0] - (\sigma^2/2i\hbar^2\omega M)[X_0^0, \hat{p}_0] \\
&= -(\sigma^2/\hbar^2)\hat{t} - (\sigma^3/2i\hbar^2\omega M)[X_0^0, X_0^{N+1}] \\
&= -(\sigma^2/\hbar^2)\hat{t} - (\sigma^2/2\hbar^2)\hat{t} \\
&= -(3\sigma^2/2\hbar^2)\hat{t}.
\end{aligned} \tag{50}$$

This provides us with an interpretation of the regularization parameter σ. Equation (50) expresses the periodicity of $\hat{t}$ as a function of τ with an angular frequency Ω such that

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\hat{t} = -\Omega^2\hat{t}, \tag{51}$$

We may then solve for σ.

$$\sigma^2 = 2\hbar^2\Omega^2/3 \tag{52}$$

The singular limit then occurs as the τ-period of the system clock becomes infinite and hence its angular frequency zero. We should then compare the singular and regular systems where $\hat{t} \sim 0$ and thus the τ-acceleration of $\hat{t}$ is small. The important variable then is $\dot{\hat{t}}$ defined by (49).

$$\dot{\hat{t}} = \frac{\mathsf{d}t}{\mathsf{d}\tau}\hat{t} = \hat{r} - (\Omega^2/3\hbar\omega M)X_0^0, \tag{53}$$

When we consider the regularized representations this value needs to be near unity not only in the singular limit but away from it as well.

The regularized variables and Lie relations given (52) are

$$\begin{aligned} \hat{p}_a &= \hbar\sqrt{2/3}\Omega X_a^{N+1} \\ \hat{q}^a &= (\hbar\sqrt{2/3}\Omega/\omega M)Y_a^{N+1} \\ \hat{r} &= (2\hbar\Omega^2/3\omega M)X_{N+1}^{N+1} \end{aligned} \tag{54}$$

for $a = 0, 1, 2, \cdots N$, and

$$\begin{aligned} [\hat{q}^a, \hat{p}_b] &= +i\hbar(2\hbar\Omega^2/3\omega M)[\delta_b^a X_{N+1}^{N+1} - X_b^a] \\ [\hat{r}, \hat{p}_b] &= -(2i\hbar\Omega^2/3)q^b \\ [\hat{r}, \hat{q}^a] &= +(2i\hbar\Omega^2/3\omega^2 M^2)p_a \\ [\hat{p}_a, \hat{p}_b] &= +(i\hbar^2\Omega^2/3)Y_a^b \\ [\hat{q}^a, \hat{q}^b] &= +(i\hbar^2\Omega^2/3\omega^2 M^2)Y_a^b \end{aligned} \tag{55}$$

for a, b again ranging from 0 to N.

We close this section with a pair of observations.

1. Up to this point we have presented the Lie relations without reference to a linear representation space. Until some representation is specified Hermiticity is not well defined. However the notation for the standard generators of $\mathsf{sl}(N+2)$ is such that with respect to the particular adjoint

$$[\alpha\hat{\Lambda}_j^i]^\dagger \equiv \alpha^*\hat{\Lambda}_i^j \tag{56}$$

all of the regularized varibles are $\dagger$-Hermitian Thus their imaginary multiples are generators of $\mathsf{SU}(N+2)$.

2. We note that in the expanded coordinates no longer commute, $[\hat{q}^a, \hat{q}^b] \neq 0$. The regularization process introduces a *non-commutative geometry*. One immediate prediction is that it will be impossible to simultaneously localize the system within a given coordinate volume. This defines a minimal localization scale without directly invoking gravitation and event horizon formation.

3.5 Representations

The next stage in the regularization procedure is to select a finite dimensional representation in which the dynamicals are Hermitian and are close enough to the usual singular Heisenberg operators to account for past experimental data. It suffices to give an infinite sequence of such representations non-uniformly converging to the singular one, leaving it to experiment to decide how far out in the sequence we have to go.

We now outline this procedure for the case of the one dimensional dynamic oscillator. The expanded Lie algebra is then $\mathsf{sl}(N+2) = \mathsf{sl}(3)$. The irreducible representations for $\mathsf{sl}(3)$ are supported on the tensor algebra over the fundamental (vector) representation, $\varrho^{(1)}_{(0)} : \mathsf{sl}(3) \to \mathsf{End}(\mathbf{V})$ where $\mathbf{V} \simeq \mathbb{C}^3$. The irreducible representations $\varrho^{(\mu)}_{(\nu)}$ are the tensors over $\mathbf{V}$ corresponding to the Young diagrams in Figure 1.

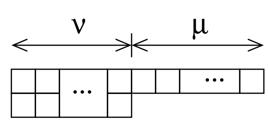

Figure 1: Young Diagram

The weight diagram of the $\varrho^{(\mu)}_{(\nu)}$ irrep is shown in Figure 2 with maximum weight $\mu\mathbf{w}_1 + \nu\mathbf{w}_2$ in a given Cartan basis.

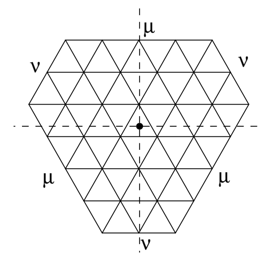

Figure 2: Representation Weights.

The normalized generators are.

$$\begin{aligned}
(1/\sigma)\hat{p}_0 &= X^2_0 \\
(M\omega/\sigma)\hat{t} &= Y^2_0 \\
(1/\sigma)\hat{p}_1 &= X^2_1 \\
(M\omega/\sigma)\hat{q}^1 &= Y^2_1 \\
(\hbar M\omega/\sigma^2)\hat{r} &= X^2_2 \\
(1/\hbar\omega)\hat{H} &= -X^1_1.
\end{aligned} \tag{57}$$

There are two more independent generators, X^1_0 and Y^1_0. The weights in the adjoint representation $\varrho^{(1)}_{(1)}$ are shown in Figure 3.

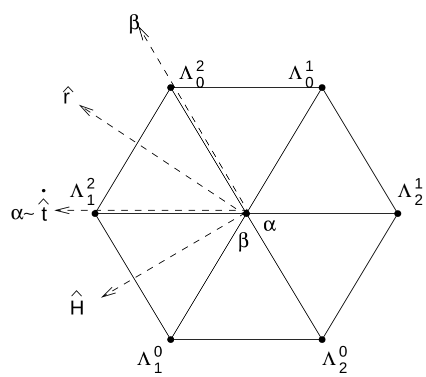

Figure 3: Adjoint Representation.

The dotted arrows in Figure 3 show the direction of increasing eigen-values for the respective elements of the Cartan algebra. The choice of Cartan subalgebra is important for determining the relationship between variables in the singular limit. We may take as a Cartan subalgebra the elements

$$\begin{aligned}\alpha &= X_2^2 - X_1^1 = (\hbar M\omega/\sigma^2)\hat{r} + (1/\hbar\omega)\hat{H} \\ \beta &= X_2^2 - X_0^0 = (2\hbar M\omega/\sigma^2)\hat{r} - (1/\hbar\omega)\hat{H}. \end{aligned} \tag{58}$$

These express the two non-commuting $\mathsf{sl}(2)$ subalgebras generated by $(\alpha, \hat{p}_1, \hat{q}^1)$ and $(\beta, \hat{t}, \hat{p}_0)$.

Note in particular that $\alpha \propto \dot{\hat{t}}$. The value of Ω is then determined by the condition $\dot{\hat{t}} \doteq 1$. It is then a simple matter to take the limit as $\mu \to \infty$ and $\nu \to \infty$ as $\sigma \propto \Omega \to 0$ to recover the singular limit.

This completes the construction of the regularization and begins the process of comparison with experiment.

CHAPTER 4

CONCLUSIONS

We give a viable theory of quantized time, perhaps the first. It respects the usual symmetries, exactly for regular symmetries like rotational invariance, and as closely as desired for singular ones like translational invariance, which are only approximate anyway. The generalization to relativistic theories will likewise preserve Lorentz invariance exactly.

It is possible to regularize the canonical Lie algebra for r dimensions, $\mathsf{h}(r)$, by a motion in an envelope algebra $\mathsf{sl}(r+2) \supset \mathsf{h}(r)$. We give such a motion. It induces the singularization $\mathsf{sl}(r+1) \to \mathsf{sl}(r) \ltimes \mathsf{h}(r)$.

We also define Lie algebras, $\boldsymbol{\mathcal{L}}_{\rm HO}(N)$ for the dynamical theory of the isotropic harmonic oscillator in N dimensional space and 1-dimensional time. These express both the commutation relations and the dynamical equations of the oscillator in in one Lie algebra, making use of the special circumstance that the dynamical equations of the harmonic oscillator are linear.

The dynamical algebra of an n-dimensional quantum oscillator is isomorphic to the Lie algebra of a stationary $N+1$-dimensional quantum system that can be interpreted as the composite of the original quantum system and a quantum clock. Their regularization is induced in a natural way by an inner motion in the special linear algebra $\mathsf{sl}(N+3)$ which defines the regularizing motion for the singular subalgebra$\mathsf{hsl}(N+1) \supset \mathsf{h}(N+1)$. This motion is the main result of the work.

As in special relativity, there are an infinity of time variables associated with an infinity of reference frames, but now they do not necessarily commute. They differ by how they split the united system into system and extra-systemic clock. It is possible that this will be a significant physical effect in the subatomic domain only. Each frame has a definite beginning and end of its time spectrum, but the end of time for one frame may be the middle of time for another.

The regular oscillator in N dimensions violates the Heisenberg uncertainty relations, the virial theorem, and the equipartition theorem. The violations are as small as desired in states near the ground state of oscillators of medium frequency, and are overwhelming in all states of extremely hard or soft oscillators.

This work extends to higher dimensions and to the dynamical theory the results already found for one dimension and the stationary theory by Mohsen Shiri-Garakani[13].

We mention the next steps in this program. Since the theory embraces both special relativity and quantum theory it has a certain amount of contact with experiment. The survival of the theory depends on its new predictions, however. These are expected in the domain of ultra-high-energy corrections to field theory. Here we have studied the regularization of a quantum mechanical theory. The regularization process developed and tested here must next be applied to the field theories of gravity and the standard model.

Nevertheless, the regularized theory implies small corrections to quantum theory and special relativity at ordinary energies, just as special relativity implied small corrections to mechanics at non-relativistic energies that were actually measured before the theory was formulated. Every experimental verification of either special relativity sets upper bounds on our new structure constants.

APPENDIX A

THE CLASSICAL LIE GROUPS

In this appendix we outline the notation we utilize for the presentation of the classic Lie groups and their Lie algebras. We utilize a colon seperator on multiple indices to represent the fact that they do not range independently but rather have symmetrization relations. Indices enclosed in square braces i.e. $[a\!:\!b\!:\!c]$ indicate total antisymmetry while parentheses indicate total symmetry.

A.1 The Orthogonal Groups

We define a basis $\Omega_{[j:k]}$ for the Lie algebra $\mathsf{so}(N;g)$ where $j,k \in \{1,2,\cdots,N\}$ and the double idex is anti-symmetric,

$$\Omega_{[j:k]} = -\Omega_{[k:j]} \tag{59}$$

The pair $[j\!:\!k]$ acts as a single index of the Lie algebra, ranging from $[1\!:\!2]$ to $[(N-1)\!:\!N]$.

We may then make use of a symmetric bilinear form $G_{(j:k)}$ to define the Lie product of two basis elements.

$$\Omega_{[j:k]} \vartriangle \Omega_{[m:n]} = G_{(k:m)}\Omega_{[j:n]} + G_{(j:n)}\Omega_{[k:m]} - G_{(j:m)}\Omega_{[k:n]} - G_{(k:n)}\Omega_{[j:m]} \tag{60}$$

This double index notation expresses an underlying *Inhomogenous-Clifford* or *Clifford-Grassmann* algebra $\boldsymbol{\mathcal{CG}}(N;g)$. This is an associative semi-graded algebra generated by single indexed elements ξ_k which have semi-grade one. The defining relations are

$$\xi_j\xi_k + \xi_k\xi_j = G_{(j:k)}\mathbf{1} \tag{61}$$

In the case where g is non-singular the algebra is a *Clifford algebra*, and in the extreme case where g is totally singular, i.e. $G_{jk} = 0$, the algebra reduces to a *Grassmann algebra*.

The grade two elements then may be identified with the generators of the special orthogonal Lie algebra $\mathsf{so}(N;G)$ with the commutator of the associative product defining the

Lie product $\triangle$.

$$\xi_j \xi_k - \xi_k \xi_j = 2\Omega_{[j:k]} \tag{62}$$

Via similarity transformations on the generators of $\boldsymbol{\mathcal{CG}}(N; g)$ the symmetric form (metric) G may be brought into anormalized diagonal form (63).

$$g \sim \begin{pmatrix} \mathbf{1}_p & & \\ & -\mathbf{1}_q & \\ & & \mathbf{0}_z \end{pmatrix} \tag{63}$$

where $p+q+z = N$. Thus the orthogonal Lie algebras and Clifford-Grassmann algebras of identical dimensional signature (p, n, z) are isomorphic. We indicate the respective algebras via the notation $\mathsf{so}(p, q, z)$ and $\boldsymbol{\mathcal{CG}}(p, q, z)$. When $z = 0$ or $z = q = 0$ they may be dropped from the notation as for example $\mathsf{so}(p) = \mathsf{so}(p, 0, 0)$ and $\boldsymbol{\mathcal{CG}}(p, q) = (p, q, 0)$.

We refer to the cases $\mathsf{so}(p, n, z)$ where $z \neq 0$ as *singular orthogonal Lie algebras* and their corresponding groups as the *singular orthogonal groups*. In the special case where $(p, q, z) = (p, q, 1)$ we have in particular the inhomogenous orthogonal Lie algebra $\mathsf{so}(p, q, 1) \equiv \mathsf{iso}(p, q)$ and group $\mathsf{SO}(p, q, 1) = \mathsf{ISO}(p, q)$.

An example relevant to Newtonian physics is the full group of Galilean relativity, $\mathsf{SO}(3, 0, 2)$. Within this group are the spatial rotations acting isomorphically on both displacements of position and of velocity. The group also contains as a normal subgroup the three dimensional translations of position (*displacements*) and of velocity (*boosts*). There is also a central time translation subgroup bringing the total dimension of the full Galilean relativity group to $3 + 3 + 3 + 1 = 10$.

$$\mathsf{SO}(3, 0, 2) \simeq \mathsf{SO}(3) \ltimes [\mathsf{P}^t(1) \times \mathsf{P}^r(3) \times \mathsf{P}^v(3)] \tag{64}$$

A second example occurs in Dirac's theory of spinor particles. The relevant singular orthogonal group is the Poincaré group $\mathsf{ISO}(3, 1) \simeq \mathsf{SO}(3, 1, 1)$. The underlying Clifford-Grassmann algebra $\boldsymbol{\mathcal{CG}}(3, 1, 1)$ provides a representation algebra for the Poincaré group. Dirac's γ-matricies may then be identified with the grade-two generators

$$\gamma_\mu \equiv \Omega_{[\mu:0]} \tag{65}$$

where μ ranges from 1 to 4 and the zero indexed generator ξ_0 corresponds to the null subspace $\{\xi_0, \xi_k\} = G_{(0:k)} = \mathbf{0}$. The remainder of the metric $G_{(\mu:\nu)}$ corresponds to the Minkowski metric $g_{\mu\nu}$ ($\mu, \nu \in \{1, 2, 3, 4\}$).

The Clifford-Grassmann algebras have both an integer semi-grade and a proper $\mathbb{Z}_2$ grading defined by the even-odd pairity of the semi-grade. The even elements of $\boldsymbol{\mathcal{CG}}$ form a sub-algebra isomorphic to a smaller Clifford-Grassmann algebra. The new semi-grade of the even sub-algebra is acheived by demoting one of the grade-one generators, say ξ_0 to grade zero e.g. $\xi_{[0}\xi_{\mu]} \to \tilde{\xi}_\mu$.

Repeated reduction to the even subalgebras in this manner provided the nested orthogonal subgroup representations for the sequence of subgroups (66).

$$\mathsf{SO}(2) \subset \mathsf{SO}(3) \subset \cdots \subset \mathsf{SO}(N) \tag{66}$$

Which may be generalized in the obvious fashion to indefinite and singular cases.

A.2 The Symplectic Groups

We define a basis $\Sigma_{(i:k)}$ $i, k \in \{1, 2, \cdots, N\}$ for the Lie algebra $\mathsf{sp}(N; J)$ where the double index is symmetric,

$$\Sigma_{(i:k)} = \Sigma_{(k:i)} \tag{67}$$

Again these pairs $(i\!:\!k)$ should be considered as a single index ranging from $(1:1)$ to $(N:N)$. We then define the Lie product utilizing an anti-symmetric bilinear (symplectic) form $J_{[i:k]}$.

$$\Sigma_{(i:k)} \vartriangle \Sigma_{(n:m)} = J_{[k:n]}\Sigma_{(i:m)} + J_{[i:m]}\Sigma_{(k:n)} + J_{[i:n]}\Sigma_{(k:m)} + J_{[k:m]}\Sigma_{(i:n)} \tag{68}$$

By allowing cases where J is singular we define an extended class of *singular symplectic Lie algebras* $\mathsf{sp}(2M, z)$ and their corresponding *singular symplectic groups* $\mathsf{Sp}(2M, z)$.

By a similarity transformation we may bring the symplectic metric into a standard form:

$$J_{[i:k]} \sim \begin{pmatrix} & -\mathbf{1}_M & \\ +\mathbf{1}_M & & \\ & & \mathbf{0}_z \end{pmatrix} \tag{69}$$

where $\mathbf{1}_M$ is the $M \times M$ unit matrix, and $\mathbf{0}_z$ the $z \times z$ null matrix. The symplectic groups with same dimension and null signature are thus isomorphic.

As in the previous section we find an underlying associative representation algebra $\boldsymbol{\mathcal{AH}}(2M, z)$, the *inhomogenous Heisenberg algebra*. $\boldsymbol{\mathcal{AH}}(2M, z)$ is constructed from $2M + z$ generators ξ_k satisfying the relations (70).

$$\xi_i \xi_k - \xi_k \xi_i = J_{ik} \mathbf{1} \tag{70}$$

This is again a semi-graded algebra with the generators ξ_k being assigned grade one. The grade two elements (71) are identified with the generators of the symplectic Lie algebra $\mathsf{sp}(2M, z)$.

$$\{\xi_i, \xi_k\} \equiv \xi_i \xi_k + \xi_k \xi_i = 2\Sigma_{(i:k)} \tag{71}$$

In the cases where $z = 0$ the algebra is in fact a standard *Heisenberg algebra* while when the symplectic form is totally singular $M = 0$ the result is the Abelian algebra of polynomials in ξ_k.

There is again a recursive embedding we may observe by considering that again there is an 2-grading defined by the even-odd pairity of the semi-grade. The even elements form a sub-algebra isomorphic to a smaller inhomogenous Heisenberg algebra with new semi-grade obtained by reducing the grade of one of the generators to zero, e.g. $\Sigma_{(0:k)} \equiv \xi_{(0}\xi_{k)} \to \tilde{\xi}_k$.

A.3 The Linear and Groups

Given the strong parallels between the symplectic (Cartan's C series) and orthogonal groups (Cartan's B and D series) we may seek a similar format for the presentation of the linear groups (Cartan's A series). The use of respectively symmetric and anti-symmetric forms to define the orthogonal and symplectic groups are however means of embedding these Lie groups as subgroups of an enveloping special linear group. The single indices reflect the vector basis for the representations corresponding to the injective Lie homomorphisms

$$\mathsf{so}(p, q, z) \xrightarrow{\rho_v} \mathsf{sl}(p + q + z) \tag{72}$$

$$^{(}2M, z) \xrightarrow{\rho_v} \mathsf{sl}(2M + z) \tag{73}$$

We may reverse the defining sequence and embed the unitary and linear groups and Lie algebras within larger orthogonal or symplectic algebras, e.g. $\mathsf{sl}(n) \to \mathsf{so}(n,n)$ and $\mathsf{su}(n) \to^{(} 2n)$. Although these constructions are informative we shall here construct the linear and unitary groups directly.

We construct the special linear Lie algebra $\mathsf{sl}(N;\mathbb{F})$ where $\mathbb{F}$ is one of the number fields $\mathbb{R}$,$\mathbb{C}$, or $\mathbb{H}$ by defining a generator basis Λ^i_j where i,j range from $1 \to N$, subject only to the condition

$$\sum_k \Lambda^k_k = 0 \tag{74}$$

In terms of a vector representation ρ_v mapping the generator Λ^i_j to an operator on a linear space $\mathbf{V}$ with basis $|k\rangle$ the specific operator identification is given by (75) below.

$$\Lambda^i_j \xrightarrow{\rho_v} |i\rangle\langle j| - \frac{1}{N}\delta^i_j\mathbf{1} \tag{75}$$

where $\mathbf{1} = \sum_k |k\rangle\langle k|$ and $\langle j \mid k\rangle = \delta^k_j$.

We emphasize the distinction between the Lie element Λ^i_j and any one of its many representations, e.g. (75). It is the representation on which a trace is defined. The index pair $\left\{{i \atop j}\right\}$ associated with the generator Λ^i_j should be considered as a single index in the set $\{\left\{{1 \atop 1}\right\}, \left\{{1 \atop 2}\right\}, \cdots, \left\{{N \atop N}\right\}\} \setminus \left\{{k \atop k}\right\}$ for some fixed k.

The Lie product is then

$$\Lambda^i_j \vartriangle \Lambda^n_m = \delta^n_j\Lambda^i_m - \delta^i_m\Lambda^n_j \equiv C_{\left\{{i \atop j}\right\}}{}^{\left\{{r \atop s}\right\}}{}_{\left\{{n \atop m}\right\}}\Lambda^r_s \tag{76}$$

where the structure coefficients are

$$C_{\left\{{i \atop j}\right\}}{}^{\left\{{r \atop s}\right\}}{}_{\left\{{n \atop m}\right\}} \equiv \delta^n_j\delta^i_r\delta^s_m - \delta^i_m\delta^n_r\delta^s_j. \tag{77}$$

APPENDIX B

INHOMOGENOUS LIE GROUPS

We include in this appendix some generalizations of the inhomogenous extensions of simple Lie groups.

B.1 Singularizations Preserving Structures

We now define a more general class of singularizations that preserve compound subgroup structures such as

$$\mathsf{SL}(\mathbf{V}) \times \mathsf{SL}(\mathbf{V}') \subset \mathsf{SL}(\mathbf{V} \oplus \mathbf{V}'). \tag{78}$$

For this purpose we define an inhomogenous version $\boxtimes$ of the Cartesian product $\times$ of two groups in the same Cartan class. This inhomogenous product is an example of a *twisted product*. We begin with the linear groups, as the other classical groups are defined as subgroups.

Definition 3 (Inhomogenous Product). *The* inhomogenous product $\mathsf{SL}(\mathbf{V}) \boxtimes \mathsf{SL}(\mathbf{V}')$ *of two special linear groups is*

$$\mathsf{SL}(\mathbf{V}) \boxtimes \mathsf{SL}(\mathbf{V}') = \big(\mathsf{SL}(\mathbf{V}) \times \mathsf{SL}(\mathbf{V}')\big) \ltimes \mathsf{H}(\mathbf{V}' \otimes \mathbf{V}^*) \tag{79}$$

If $\mathbf{V}$ *and* $\mathbf{V}'$ *are also both* Hermitian, quadratic, *or* symplectic *spaces then we define the corresponding inhomogenous products of the respective* unitary, orthogonal, *or* symplectic *groups by:*

$$\begin{aligned} \mathsf{SO}(\mathbf{V}) \boxtimes \mathsf{SO}(\mathbf{V}') &= \big(\mathsf{SU}(\mathbf{V}) \times \mathsf{SU}(\mathbf{V}')\big) \ltimes \mathsf{UHeis}(\mathbf{V}' \otimes \mathbf{V}^*) \\ \mathsf{SO}(\mathbf{V}) \boxtimes \mathsf{SO}(\mathbf{V}') &= \big(\mathsf{SO}(\mathbf{V}) \times \mathsf{SO}(\mathbf{V}')\big) \ltimes \mathsf{P}(\mathbf{V}' \otimes \mathbf{V}^*) \\ \mathsf{Sp}(\mathbf{V}) \boxtimes \mathsf{Sp}(\mathbf{V}') &= \big(\mathsf{Sp}(\mathbf{V}) \times \mathsf{Sp}(\mathbf{V}')\big) \ltimes \mathsf{P}(\mathbf{V}' \otimes \mathbf{V}^*) \end{aligned} \tag{80}$$

We extend this definition to multiple products:

Definition 4 (Multiple Inhomogenous Products). *We define the inhomogenous product of a sequence of special linear groups by:*

$$\boxtimes_{k=1}^{N} \mathsf{SL}(\mathbf{V}_k) \equiv \big[\bigtimes_{k=1}^{N} \mathsf{SL}(\mathbf{V}_k)\big] \ltimes \big[\bigtimes_{j=1,k=j+1}^{N} \mathsf{H}(\mathbf{V}_j \otimes \mathbf{V}_k^*)\big]. \tag{81}$$

If the spaces in a sequence are all *Hermitian*, *quadratic*, or *symplectic*, then the corresponding multiple inhomogenous products of the respective *unitary*, *orthogonal* or *symplectic* subgroups are defined by a similar generalization of the single products.

Theorem 1. *The dimension of the inhomogenous product of a sequence of simple invariance groups over* linear,Hermitian,quadratic *or* symplectic *spaces is equal to the dimension of the simple invariance group over the tensor sum of these spaces:*

$$\begin{aligned}
\mathtt{Dim}[\mathsf{SL}(\mathbf{V}_1) \boxtimes \mathsf{SL}(\mathbf{V}_2) \boxtimes \cdots] &= \mathtt{Dim}[\mathsf{SL}(\mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots)] \\
\mathtt{Dim}[\mathsf{SU}(\mathbf{V}_1) \boxtimes \mathsf{SU}(\mathbf{V}_2) \boxtimes \cdots] &= \mathtt{Dim}[\mathsf{SU}(\mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots)] \\
\mathtt{Dim}[\mathsf{SO}(\mathbf{V}_1) \boxtimes \mathsf{SO}(\mathbf{V}_2) \boxtimes \cdots] &= \mathtt{Dim}[\mathsf{SO}(\mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots)] \\
\mathtt{Dim}[\mathsf{Sp}(\mathbf{V}_1) \boxtimes \mathsf{Sp}(\mathbf{V}_2) \boxtimes \cdots] &= \mathtt{Dim}[\mathsf{Sp}(\mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots)]
\end{aligned} \tag{82}$$

This result is straightforward and in fact was the clue to the construction of the inhomogenous product.

We outline the proof by giving the differences in the dimension of the groups over tensor sums of spaces and the product of groups over each space. For the orthogonal and symplectic cases:

$$(\sum_k d_k)(\sum_k d_k \pm 1)/2 - \sum_k d_k(d_k \pm 1)/2 = \sum_{j>k} d_j d_k \tag{83}$$

and note that $\mathtt{Dim}(\mathsf{P}(\mathbf{V}_j \otimes \mathbf{V}_k^*) = \mathtt{Dim}(\mathbf{V}_j)\mathtt{Dim}(\mathbf{V}_k)$.

For the linear and unitary cases,

$$(\sum_k d_k)^2 - 1 - \sum_k (d_k^2 - 1) = \sum_{j<k} (2d_j d_k + 1) \tag{84}$$

and note that $\mathtt{Dim}(\mathsf{H}(\mathbf{V})) = \mathtt{Dim}(\mathsf{UHeis}(\mathbf{V})) = 2d + 1$ where $\mathtt{Dim}(\mathbf{V}) = d$.

We hypothesize but leave unproven the following assertion.

Hypothesis 1. *The inhomogenous product group iprod*$_{k=1}^{N}\mathsf{SL}(\mathbf{V}_k)$ *is a singular limit of the of the simple Lie group* $\mathsf{SL}\left(\bigoplus_{k=1}^{N} \mathbf{V}_k\right)$ *under a singularizing homotopy.*

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